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Math 451: Intro. to General Topology — HOMEWORK 6 — **Due: Apr 10, 23:59**

1. Let X be a set, $\{Y_i\}_{i\in I}$ be a family of topological spaces, and $\{f_i : X \to Y_i\}_{i\in I}$ be a family of functions. Equip X with the topology generated by all the f_i, $i \in I$. (For example, when $X := \prod_{i\in I} X_i$ and $f_i := \text{proj}_i$, the topology generated by all the f_i is exactly the product topology.) Prove:

(a) For each sequence $(x_n) \subseteq X$ and $x \in X$, we have $\lim_{n\to\infty} x_n = x$ in the topology of X if and only if for each $i \in I$, $\lim_{n\to\infty} f_i(x_n) = f_i(x)$ in the topology of Y_i.

(b) Let Z be a topological space. A function $g : Z \to X$ is continuous if and only if for each $i \in I$, the function $f_i \circ g : Z \to Y_i$ is continuous.

2. Let X be a topological space, Y be a set, and $f : X \to Y$.

(a) Define the largest topology $\mathcal{T}$ on Y with respect to which f is continuous, i.e. if $\mathcal{T}'$ is any other topology on Y making f continuous then $\mathcal{T}' \subseteq \mathcal{T}$. Prove your answer.

(b) [*Optional*] When Y is the quotient X/E by some equivalence relation E on X and f is the quotient map $X \to X/E$, then this topology $\mathcal{T}$ on X/E (as in the previous part) is called the **quotient topology**. Now let $X := \mathbb{R}$ and E be the coset equivalence relation of $\mathbb{Z} \leqslant \mathbb{R}$, i.e. $x \mathrel{E} y$ exactly when $x - y \in \mathbb{Z}$, for all $x, y \in \mathbb{R}$. Prove that $\mathbb{R}/E$ equipped with the quotient topology is homeomorphic to unit circle S^1 in $\mathbb{R}^2$.

3. Prove that in the cofinite topology on any infinite set X, every subset $Y \subseteq X$ is compact.

4. [*Optional*] Kőnig's lemma states that every infinite finitely-branching tree has an infinite branch. Prove this and use it to derive a proof of the compactness of $2^{\mathbb{N}}$ different from the one given in lecture.

Definition. A metric space (X, d) is called **totally bounded** if for each $\varepsilon > 0$ there is a finite cover of X with sets of diameter $\leqslant \varepsilon$.

5. Prove directly from the definition that in the usual metric, $2^{\mathbb{N}}$ is totally bounded but $\mathbb{N}^{\mathbb{N}}$ is not.

MORE QUESTIONS TO BE ADDED.